EVOLUTION OF THERMAL EXCITATIONS IN A MEDIUM WITH MEMORY

I. A. Novikov

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The effects of thermal memory on the propagation of delta-function and unit-step function temperature pulses in a semiinfinite medium are discussed. It is shown that an amplifying thermal medium is possible.

The conduction of heat in a medium with fading thermal memory of the Nunziato [1] form is governed by a pair of linearized integral relations and the conservation law for the internal energy:

$$q(x,\ t)=\lambda_0\left[-\lambda_1(0)\nabla u(x,\ t)-\int\limits_0^\infty \frac{d\lambda_1(\tau)}{d\tau}\nabla u(x,\ t-\tau)\,d\tau\right];$$

$$e(x,\ t)=e_0+\frac{\lambda_0}{a_0}\left[c_1(0)u(x,\ t)+\int\limits_0^\infty \frac{dc_1(\tau)}{d\tau}u(x,\ t-\tau)\,d\tau\right];\tag{1}$$

$$\frac{\partial e}{\partial t}=-\operatorname{div}q+b(x,\ t);\ x>0;\ t>0.$$

Here, $u(x,\ t) = T(x,\ t) - T(x,\ 0)$ and we take $u(x,\ 0) = 0$ and $b = 0$. The boundary condition $u(0,\ t) = u_0(t)$ is specified at $x = 0$. We take the Laplace transform of (1) with respect to t and x. Then, taking into account the initial condition and boundary condition, we find the solution for an arbitrary damping thermal medium [2]:

$$u(x,\ t)=\int\limits_0^t u(0,\ t-\tau)\,a\left(\tau,\ \frac{x}{\sqrt{a_0}}\right)d\tau;\ K_0(p)=\sqrt{pC_1(p)/\Lambda_1(p)};$$

$$q(x,\ t)=\int\limits_0^t q(0,\ t-\tau)\,a\left(\tau,\ \frac{x}{\sqrt{a_0}}\right)d\tau;\tag{2}$$

$$a\left(t,\ \frac{x}{\sqrt{a_0}}\right)=L^{-1}\exp\left[-\frac{x}{\sqrt{a_0}}K_0(p)\right];$$

$$u(0,\ t)=u_0(t);\ q(0,\ t)=\frac{\lambda_0}{\sqrt{a_0}}L^{-1}\left[\frac{K_0(p)}{S(p)}U_0(p)\right].$$

The expansion of the relaxation kernel in a Taylor series in powers of t gives an approximate solution for small times to an arbitrary order of approximation N (see [2]).

It is of interest to consider an idealized problem where a delta-function pulse originating at the boundary [$u(0,\ t) = \delta(t)$] propagates in the medium. It follows from (2) that

$$u(x,\ t)=a\left(t,\ \frac{x}{\sqrt{a_0}}\right);\tag{3}$$

$$q(x,\ t)=\int\limits_0^t q(0,\ t-\tau)\,a\left(\tau,\ \frac{x}{\sqrt{a_0}}\right)d\tau;\ q(0,\ t)=\frac{\lambda_0}{\sqrt{a_0}}L^{-1}\left[\frac{K_0(p)}{S(p)}\right].$$

In a Fourier medium, a power series expansion in t [2, 3] can be used in (3) for small values of the time. Then we have

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$$u(x, t)=\sum_{l=1}^{N}\frac{1}{l!}\left(\frac{-x}{\sqrt{a_0}}\right)^l\frac{d_1x}{2\sqrt{a_0\pi}}\int_0^t \exp\left[-\frac{d_1x^2}{4a_0(t-\xi)}-\right.$$

$$\left.-\frac{d_0(t-\xi)}{d_1}\right][k_N^{[l]}(\xi)+o(\xi^{N-1})]\frac{d\xi}{\sqrt{(t-\xi)^3}}+\frac{\sqrt{d_1}x}{2\sqrt{a_0\pi t^3}}\exp\left(-\frac{d_1x^2}{4a_0t}-\frac{d_0t}{d_1}\right); \tag{4}$$

$$q(x, t)=\eta\int_0^t\left[\frac{1}{\sqrt{\pi(t-\xi)^3}}+\sum_{r=1}^{N}\frac{\left(\prod_{i=0}^{r-1}\left(\frac{1}{2}-i\right)\right)}{r!}\sum_{m=r}^{N}\frac{s_m^{[r]}(t-\xi)^{m-3/2}}{\Gamma(m-1/2)}+o((t-\xi)^{N-1})\right]u(x, \xi)\,d\xi;$$

$$\eta=\frac{\lambda_0}{\sqrt{a_0}}\sqrt{\lambda_1(0)c_1(0)};$$

$$d_1=c_1(0)/\lambda_1(0);\ d_0=d_1\left[\frac{c_1^{(1)}(0)}{c_1(0)}-\frac{\lambda_1^{(1)}(0)}{\lambda_1(0)}\right],$$

where the rest of the notation is taken from [2]. It is seen from (4) that a pulse initially localized on the boundary spreads out as it propagates to the right. Its spatial distribution widens and the maximum temperature decreases in time. Considering only the first-order approximation (the term outside of the integral), we see that

$$x_{\max}=\sqrt{2a_0t/d_1};\ u_{\max}=\frac{1}{\sqrt{2\pi}\,t}\exp\left(-\frac{1}{2}-\frac{d_0t}{d_1}\right);\ \begin{matrix}d_1>0;\\ d_0\geqslant 0.\end{matrix} \tag{5}$$

The higher-order approximations do not change the qualitative picture given by the first-order approximation. Because $u(x, t)=a(t, x/\sqrt{a_0})\underset{x\to 0}{\to}\delta(t)$, the thermal flux at the wall at the initial instant of time is unbounded and for small t behaves as $q(0, t) = \eta\pi^{-1/2}t^{-3/2} + s_1\cdot(\pi t)^{-1/2} + o(1)$. When $c_1(0) = \lambda_1(0) = 1$ and $d_0 = s_m = 0$, we have the solution for a purely Fourier medium.

In a Maxwell medium the solution for small times has the form

$$u(x, t)=\exp(-d_1\gamma t)\left[\delta\left(t-\frac{x}{w}\right)+\frac{b_0x}{w}H\left(t-\frac{x}{w}\right)\frac{I_1\left(b_0\sqrt{t^2-\frac{x^2}{w^2}}\right)}{\sqrt{t^2-x^2/w^2}}\right]+\sum_{l=1}^{N}\frac{1}{l!}\left(\frac{-x}{\sqrt{a_0}}\right)^l H\left(t-\frac{x}{w}\right)\times$$

$$\times\left[\exp\left(-\frac{d_1\gamma x}{w}\right)k_N^{[l]}\left(t-\frac{x}{w}\right)+o\left(\left(t-\frac{x}{w}\right)^{N-1}\right)+\right.$$

$$\left.+\frac{b_0x}{w}\int_0^{t-\frac{x}{w}}\frac{I_1\left(b_0\sqrt{(t-\xi)^2-\frac{x^2}{w^2}}\right)}{\sqrt{(t-\xi)^2-x^2/w^2}}k_N^{[l]}(\xi)\exp(-d_1\gamma(t-\xi))\,d\xi\right]; \tag{6}$$

$$q(x, t)=\eta_0\int_0^t\left[\delta(t-\tau)+\sum_{r=1}^{N}\frac{\left(\prod_{i=0}^{r-1}\left(\frac{1}{2}-i\right)\right)}{r!}\sum_{m=r}^{N}\frac{s_{m0}^{[r]}(t-\tau)^{m-1}}{(m-1)!}+o((t-\tau)^{N-1})\right]u(x, \tau)\,d\tau;$$

$$\gamma=\frac{1}{2d_2};\ w=\sqrt{\frac{a_0}{d_2}};\ d_2=\frac{c_1(0)}{\lambda^{(1)}};\ \eta_0=\frac{\lambda_0}{\sqrt{a_0}}\sqrt{c_1(0)\lambda^{(1)}};$$

$$d_1=d_2\left[\frac{c^{(1)}}{c_1(0)}-\frac{\lambda^{(2)}}{\lambda^{(1)}}\right];\ d_0=d_2\left[\frac{c^{(2)}}{c_1(0)}-\frac{\lambda^{(3)}}{\lambda^{(1)}}\right]-d_1\frac{\lambda^{(2)}}{\lambda^{(1)}};$$

$$b_0=\frac{d_1}{2d_2}\sqrt{1-4d_0d_2/d_1^2}.$$

Like the temperature, the heat flux at the wall at the initial instant of time is a delta function. A discussion of the wave effects in the propagation of heat and the spatial wake is given below.

Equation (2) can be used to get the solution for the case of a unit temperature jump at the boundary: $u(0, t) = H(t)$. We have

$$u(x, t) = \int_0^t a(\tau, x/\sqrt{a_0})\, d\tau;$$
$$q(x, t) = \int_0^t q(0, t-\tau)\, a\left(\tau, \frac{x}{\sqrt{a_0}}\right) d\tau; \qquad (7)$$
$$q(0, t) = \frac{\lambda_0}{\sqrt{a_0}} L^{-1}\left[\frac{K_0(p)}{pS(p)}\right].$$

For small times we use a power series expansion in t and thereby obtain the solution for a Fourier medium:

$$u(x, t) = \varphi^+(t, x) + \varphi^-(t, x) + \sum_{l=1}^{N} \frac{1}{l!}\left(\frac{-x}{\sqrt{a_0}}\right)^l \int_0^t [\varphi^+(t-\xi, x) + \varphi^-(t-\xi, x)]\,[k_N^{[l]}(\xi) + o(\xi^{N-1})]\, d\xi;$$
$$\varphi^{\pm}(t, x) = \frac{1}{2} \exp\left(\pm\sqrt{\frac{d_0}{a_0}}\, x\right) \operatorname{erfc}\left(\frac{\sqrt{d_1}\, x}{2\sqrt{a_0 t}} \pm \sqrt{\frac{d_0 t}{d_1}}\right); \qquad (8)$$
$$q(x, t) = \eta\left[\frac{1}{\sqrt{\pi t}} + \sum_{r=1}^{N} \frac{\left(\prod_{i=0}^{r-1}\left(\frac{1}{2} - i\right)\right)}{r!} \sum_{m=r}^{N} \frac{s_m^{[r]}\, t^{m-1/2}}{\Gamma(m+1/2)} + o(t^{N-1})\right].$$

The terms outside of the integrals in u(x, t) and q(x, t) define the first-order approximation obtained in [3].

For a Maxwell medium the solution for a unit temperature jump at small t has the form

$$u(x, t) = \exp\left(-d_1\gamma\frac{x}{w}\right) H\left(t - \frac{x}{w}\right)\left[\frac{b_0 x}{w} \int_0^{t-\frac{x}{w}} \frac{I_1\left(b_0\sqrt{\left(\tau + \frac{x}{w}\right)^2 - \frac{x^2}{w^2}}\right)}{\sqrt{\left(\tau + \frac{x}{w}\right)^2 - \frac{x^2}{w^2}}} \exp(-d_1\gamma\tau)\, d\tau + 1\right] +$$
$$+ \sum_{l=1}^{N} \frac{1}{l!}\left(\frac{-x}{\sqrt{a_0}}\right)^l \exp\left(-d_1\gamma\frac{x}{w}\right) H\left(t - \frac{x}{w}\right)\left[\int_0^{t-\frac{x}{w}} k_N^{[l]}(\tau)\, d\tau +\right.$$
$$\left. + o\left(\left(t - \frac{x}{w}\right)^N\right) + \frac{b_0 x}{w} \int_0^{t-\frac{x}{w}} d\tau \int_0^{\tau} \frac{I_1\left(b_0\sqrt{\left(\tau + \frac{x}{w} - \xi\right)^2 - \frac{x^2}{w^2}}\right)}{\sqrt{\left(\tau + \frac{x}{w} - \xi\right)^2 - \frac{x^2}{w^2}}} \exp(-d_1\gamma(\tau - \xi))\, k_N^{[l]}(\xi)\, d\xi\right];$$
$$q(0, t) = \eta_0\left[1 + \sum_{r=1}^{N} \frac{\left(\prod_{i=0}^{r-1}\left(\frac{1}{2} - i\right)\right)}{r!} \sum_{m=r}^{N} \frac{s_{m0}^{[r]}\, t^m}{m!} + o(t^N)\right]. \qquad (9)$$

The first two terms in u(x, t) define the first-order approximation of [3]; the other terms give higher-order approximations which do not change the wave nature of the heat propagation.

Equation (2) shows that in the first two boundary-value problems for similar thermal functions, the memory of the medium comes in only through the function $a(t, x/\sqrt{a_0})$, which can behave in different ways. For example, in a Maxwell medium for $b_0 > 0$, the function $a(t, x/\sqrt{a_0})$ has a delta-function part and an aperiodic spatial wake which leads to propagation from a boundary inside the body of attenuated, undistorted thermal signal and also leads to the appearance of a rapidly growing distorted diffusive heat signal. When $b_0 = 0$, the function $a(t, x/\sqrt{a_0})$ is an attenuated delta function for small times and this described the propagation of a damped, undistorted thermal signal inside the body. For imaginary $b_0(d_1^2 <$

$4d_0d_2$) and small times the diffusive wake in $a(t, x/\sqrt{a_0})$ changes into a term with a spatially oscillating wake [3], and this leads to another type of distorted thermal signal. In a medium with fading memory, the temperature at large times approaches that of an ordinary Fourier medium. We note that wave effects in a Maxwell medium suggest a different kind of problem: the propagation of a thermal signal either without distortion or with a given level of distortion and attenuation over a given distance. Finally, it would be necessary to create a medium with thermal memory having the right properties, particularly small damping.

At each point of a Maxwell medium with thermal memory, we put a source with strength proportional to the temperature ($b = b_1 u$). All the above formulas remain valid if we replace the expression for $K_0(p)$ as follows:

$$K_0(p) = \sqrt{\left[pC_1(p) - \frac{a_0 b_1}{\lambda_0 p}\right] \Big/ \Lambda_1(p)} \qquad (10)$$

and we replace $c^{(1)}$ by $c^{(1)} - a_0 b_1/\lambda_0$. For $b_1 > 0$, the function $K_0(p)$ describes a solution which is unbounded for $t \to \infty$ $[K_0(p) \to \sqrt{p - a_0 b_1/\lambda_0}]$. Thus, this case describes in principle a weakly absorbing or even amplifying thermal medium $[\tilde{d}_1 = d_1 - d_2 a_0 b_1/(\lambda_0 c_1(0)),\ \tilde{d}_1 < 0]$, and this would allow the spatial transport of a thermal signal with amplification. In practice such sources can be created in the medium by using an external field such as an electric current, or electrostatic, magnetostatic, or electromagnetic fields, which act because of the dependence of the physical properties of the medium on temperature. Because of the strong attenuation of the wave part of the solution, the dependence of the physical properties of the medium on temperature must be very sharp. Therefore, it is convenient to use the phase transition region where sharp changes in the properties of the medium occur. The action of the external fields can be increased with the help of amplifiers or the simultaneous application of several external fields.

We note that (10) is also correct for a Fourier medium; in this case, $\tilde{d}_0 = d_0 - d_1 a_0 b_1/(\lambda_0 c_1(0))$. If $\tilde{d}_0 < 0$, then transport of a thermal signal with amplification can occur. It follows from (5) that in the propagation of a delta-function pulse, u_{max} will start increasing after a certain time interval. Application of thermal materials with memory and small damping properties opens up new possibilities in the transport of heat; therefore, it is of great interest to try to find (or create) materials having these thermal memory effects.

NOTATION

λ_0, equilibrium thermal conductivity; ρ_0, density; c_0, equilibrium heat capacity; e, internal energy; e_0, initial internal energy; T, temperature; $a_0 = \lambda_0/\rho_0 c_0$, equilibrium thermal diffusivity; $\lambda(t)$, $c(t)$, relaxation kernels for heat flux and internal energy; $\lambda_1(t) = \lambda(t)/\lambda_0$, $c_1(t) = c(t)/\rho_0 c_0$, dimensionless relaxation kernels for heat flux and internal energy; L, L^{-1}, operators for the Laplace transform and its inverse; p, Laplace transform variable; H(t), Heaviside unit step function; $u_0(t)$, ambient temperature; $S(p) = [p\Lambda_1(p)]^{-1}$; $\delta(t)$, Dirac delta function; $c^{(n)} = d^n c_1(0)/dt^n$; $\lambda^{(n)} = d^n \lambda_1(0)/dt^n$; $s_1 = c_1^{(1)}/c_1(0) + \lambda_1^{(1)}/\lambda_1(0)$; I_1, modified Bessel function of the first kind of order one.

LITERATURE CITED

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